Analysis on groups

Problem Set 1 — 22 February 2018

This problem set is not representative of "Analysis on groups". We just want to collect some useful tools from functional analysis. Then we will move on!

Exercise 1
Make sure that you know the following notions, using, if needed, the library and Wikipedia:

– the dual V^* of a normed vector space V,
– the weak topology on V and the weak-* topology on V^*,
– the norm of a linear map between normed vector spaces,
– "the" Hahn–Banach theorem for normed spaces (see e.g. the section "important consequences" on the English Wikipedia).

Exercise 2
Let V be a Banach space. Define $\iota \colon V \to V^{**}$ by $\iota(v)(f) = f(v)$. Prove:

(i) ι is well-defined, linear and isometric.

(ii) ι is a homeomorphism onto its image when V has the weak topology and V^{**} the weak-* topology.

(iii) $\iota(V)$ is weak-* dense in V^{**}.

(iv) If $\lambda \in V^{**}$ is weak-* continuous as a map $V^* \to \mathbf{R}$, then $\lambda \in \iota(V)$.

Exercise 3
Let V, W be Banach spaces.

(i) Given a continuous linear map $\alpha \colon V \to W$, define $\alpha^* \colon W^* \to V^*$ and check that $\|\alpha^*\| = \|\alpha\|$.

(ii) Let $\beta \colon W^* \to V^*$ be a continuous linear map. Prove that $\beta = \alpha^*$ for some $\alpha \colon V \to W$ if and only if β is continuous for the weak-* topologies. *Hint: use 2(iv).*

Exercise 4
Let X be a set. Given $a \in \ell^1(X)$ and $b \in \ell^\infty(X)$, we can define the number $\langle a, b\rangle = \sum_{x \in X} a(x)b(x)$. (BTW: why?) Use this to construct an isometric isomorphism between $\ell^\infty(X)$ and $\ell^1(X)^*$.

Additional question: why does it not *similarily identify $\ell^1(X)$ with $\ell^\infty(X)^*$?*