

Analysis on groups

Problem Set 1

23 February 2017

Exercise 1

Make sure that you know the following notions, using, if needed, the library and Wikipedia:

- the dual V^* of a normed vector space V ,
- the weak topology on V and the weak-* topology on V^* ,
- the norm of a linear map between normed vector spaces,
- “the” Hahn–Banach theorem for normed spaces (see e.g. the section “important consequences” on the English Wikipedia).

Exercise 2

Let V be a Banach space. Define $\iota: V \rightarrow V^{**}$ by $\iota(v)(f) = f(v)$. Prove:

- ι is well-defined, linear and isometric.
- ι is a homeomorphism onto its image when V has the weak topology and V^{**} the weak-* topology.
- $\iota(V)$ is weak-* dense in V^{**} .
- If $\lambda \in V^{**}$ is weak-* continuous as a map $V^* \rightarrow \mathbf{R}$, then $\lambda \in \iota(V)$.

Exercise 3

Let V, W be Banach spaces.

- Given a continuous linear map $\alpha: V \rightarrow W$, define $\alpha^*: W^* \rightarrow V^*$ and check that $\|\alpha^*\| = \|\alpha\|$.
- Let $\beta: W^* \rightarrow V^*$ be a continuous linear map. Prove that $\beta = \alpha^*$ for some $\alpha: V \rightarrow W$ if and only if β is continuous for the weak-* topologies. *Hint: use 2(iv).*

Exercise 4

Let X be a set. Given $a \in \ell^1(X)$ and $b \in \ell^\infty(X)$, we can define the number $\langle a, b \rangle = \sum_{x \in X} a(x)b(x)$. (BTW: why?) Use this to construct an isometric isomorphism between $\ell^\infty(X)$ and $\ell^1(X)^*$.

Additional question: why does it not similarly identify $\ell^1(X)$ with $\ell^\infty(X)^$?*