Analysis on groups

Problem Set 1 21 February 2019

This problem set is not representative of “Analysis on groups”. We just want to collect some useful tools from functional analysis. Then we will move on!

Exercise -1
Read http://en.wikipedia.org/wiki/Net_(mathematics) or use the library to understand the concept of nets, also known as generalized sequences. Understand the connection to open and closed sets, to continuous maps, to compactness. In particular, make sure to understand sub-nets.

Exercise 0
Make sure that you know the following notions, using, if needed, the library and Wikipedia:
– the dual $V^*$ of a normed vector space $V$,
– the weak topology on $V$ and the weak-* topology on $V^*$,
– the norm of a linear map between normed vector spaces,
– “the” Hahn–Banach theorem for normed spaces (see e.g. the section “important consequences” on the English Wikipedia).

Exercise 1
Let $V$ be a Banach space. Define $\iota: V \to V^{**}$ by $\iota(v)(f) = f(v)$. Prove:
(i) $\iota$ is well-defined, linear and isometric.
(ii) $\iota$ is a homeomorphism onto its image when $V$ has the weak topology and $V^{**}$ the weak-* topology.
(iii) $\iota(V)$ is weak-* dense in $V^{**}$.
(iv) If $\lambda \in V^{**}$ is weak-* continuous as a map $V^* \to \mathbb{R}$, then $\lambda \in \iota(V)$.

Exercise 2
Let $V, W$ be Banach spaces.
(i) Given a continuous linear map $\alpha: V \to W$, define $\alpha^*: W^* \to V^*$ and check that $\|\alpha^*\| = \|\alpha\|$.
(ii) Let $\beta: W^* \to V^*$ be a continuous linear map. Prove that $\beta = \alpha^*$ for some $\alpha: V \to W$ if and only if $\beta$ is continuous for the weak-* topologies. Hint: use Ex. 1(iv).

Exercise 3
Let $X$ be a set. Given $a \in \ell^1(X)$ and $b \in \ell^\infty(X)$, we can define the number $\langle a, b \rangle = \sum_{x \in X} a(x)b(x)$. (BTW: why?) Use this to construct an isometric isomorphism between $\ell^\infty(X)$ and $\ell^1(X)^*$. Additional question: why does it not similarly identify $\ell^1(X)$ with $\ell^\infty(X)^*$ ?